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ELEMENTARY GEOMETRY.

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ELEMENTARY GEOMETRY

PART I.

ANGLES, PARALLELS, TRIANGLES, EQUIVALENT
FIGURES, WITH THE APPLICATION TO
PROBLEMS.

COMPILED BY

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1868

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PREFACE.

THE lately published Report of the Schools Inquiry Commission has given an immediate importance to the question whether Euclid's Elements is the proper text-book for teaching Geometry to beginners. Euclid's recognised and acknowledged faults as a system of Geometry, and as a specimen of analysed reasoning, are of slight importance compared with others of greater magnitude. The real objections to Euclid as a text-book are his artificiality, the invariably syllogistic form of his reasoning, the length of his demonstrations, and his unsuggestiveness.

As to the first, he aimed, not at unfolding Geometry as a science, but at shewing on how few axioms and postulates the whole could be made to depend: and he has thus sacrificed, to a great extent, simplicity and naturalness in his demonstrations, without any corresponding gain in grasp or cogency. The exclusion of hypothetical constructions *may be mentioned* as a self-imposed restriction which has made the confused order of his first book necessary, without any compensating advantage. There is no real advantage in the arrangement of propositions by sequence which Euclid maintains so strictly, for nothing can be gained by excluding *any* sound method of reasoning: and if a direct proof can be found of any theorem, which naturally arises out of the data of the theorem, it is to be preferred to a circuitous proof which depends on other theorems. Thus if the Fifth Proposition can be proved independently, and on its own evidence, it is certain that the decisive bearing of the data on the conclusion will be better appreciated than it would be on Euclid's method.

Again, as his self-imposed restrictions, geometrical and logical, have made his Geometry confused in its arrangement, and unnatural and forced in the nature of his proofs, so too the detailed syllogistic form into which he has thrown all his reasonings, is a source of obscurity to beginners, and damaging to true geometrical freedom and power.

We put a boy down to his Euclid; and he reasons for the first time, a task in itself difficult enough; but we make him reason in iron fetters. There is, of course, a natural and inevitable difficulty in the task of tracing data into the inferences founded on them; the geometrical facts are new: it is new to the learner to find himself reasoning consecutively at all. If then to all this novelty we add the constant analysis into syllogisms of inferences which are obvious without this analysis, and the constant reference to general axioms and general propositions, which are no clearer in the general statement than they are in the particular instance, we make the study of Geometry unnecessarily stiff, obscure, tedious and barren. Many facts perfectly familiar to common sense and experience become strange and unrecognizable when thus expressed; and conclusions the most obvious become cloudy by the extreme detail of the proofs. And the result is, as every one knows, that boys may have worked at Euclid for years, and may yet know next to nothing of Geometry.

The length of his demonstrations is a real evil; for, when a geometrical theorem has become familiar, it is easy to analyse into a demonstration the perception that we have obtained of the certainty of the theorem, but until that familiarity is gained great length in demonstration exercises the memory more than the intelligence. Messrs. Demogeot and Montucci in their Report on English Education begin one of their chapters with the following remarkable words: "Le trait distinctif de l'enseignement des mathématiques en Angleterre c'est qu'on y fait appel plutôt à la mémoire qu'à l'intelligence de l'élève." And they remark later on, "On y trouve (in Euclid) sans doute une logique de fer, qui n'admet point de réplique; mais aussi arrive-t-on aux résultats les plus évidents par un verbiage absolument en désaccord avec nos élégantes habitudes d'une concision non moins vigoureuse que la prolixité d'Euclide." And unquestionably one result of the tediousness of Euclid is that so little knowledge of geometry is gained; so little, there is abundant evidence to prove, that our education is more marked by inferiority to other nations in this respect than in any other.

And again, unsuggestiveness is a great fault in a text-book. Euclid places all his theorems and problems on a level, without giving prominence to the master-theorems, or clearly indicating the master-methods. He has not, nor could he be expected to have, the modern felicity of nomenclature. The very names of *superposition*, *locus*, *intersection of loci*, *projection*, *comparison of triangles*, do not occur in his treatise. Hence there already exists a wide gulf between the form in which Euclid is read, and that in which he is generally taught. Unquestionably the best teachers depart largely from his words, and even from his methods. That is, they use the work of Euclid, but they would teach better without it. And this is especially true of the application to problems. Everybody recollects, even if he have not the daily experience, how unavailable for problems a boy's knowledge of Euclid generally is. Yet this is the true test of geometrical knowledge; and problems and original work ought to occupy a much larger share of a boy's time than they do at present.

On the other side there will be brought two arguments, and in general two arguments alone.

It will be urged that if Euclid is given up we shall lose the advantage of uniformity. Its place will be taken by scores of manuals of Geometry, and examinations in Geometry will become impossible. On such a point it will be difficult to persuade others that the advantage is nearly imaginary. The fact is, that Geometry when treated as a science, treated inartificially, falls into a certain order from which there can be no very wide departure; and the manuals of Geometry will not differ from one another nearly so widely as the manuals of algebra or chemistry: yet it is not difficult to examine in algebra and chemistry. The experience of the French schools and examinations is decisive on this point. I am assured that they experience no such difficulty in examining as we might imagine. It is true that by the loss of Euclid less weight will be given to bookwork, and more to problems, in elementary Geometry; but surely this will not be thought an evil. And to allow some latitude of expression is to admit different degrees of elegance, accuracy and perspicuity, and cannot fail to encourage these merits in the style of the student.

Secondly, it is commonly said that Euclid is taught not so much for the sake of his Geometry as for the sake of his Logic.

It is most important that there should be no confusion of ideas on this point.

Geometry is taught on several grounds; one of which is that it forms an excellent subject-matter for continuous reasoning and clear demonstration; and the merit of Euclid's treatment of Geometry consists in his resolutely fixing the mind on the *things* with which the argument is concerned, and never allowing the substitution of mere *symbols*. Algebra fails as a discipline because it speedily degenerates into a manipulation of symbols: in Euclid every step is planted on firm ground. Geometry treated algebraically would lose this merit altogether; but it is not proposed to treat Geometry algebraically. I contend that it is possible to present the science of Geometry in a more natural and simple order, and demonstrate the properties of figures by proofs differing in many cases from Euclid's, without being less rigorous.

And when it is urged that Euclid affords an excellent training in logic, it must not be forgotten that very heavy deductions have to be made from this argument.

For the test of logical training is the power of applying the same processes of reasoning to new matter. If a boy has learnt logic, he can use his logic. It is a mere delusion to believe that boys or men have got a logical training from the study of Euclid if they are unable to solve a problem or work a deduction. Their recollection of Euclid may be perfect, but the test of training is the *power* it has given; and if boys can produce no original work, the training is assuredly worth very little. The fact is, Euclid is a beautiful exercise in logic to a mathematician; but the mathematics must in many cases be learnt first, or the training is hopelessly out of the learner's reach.

Again, the artificial restrictions in Euclid are a serious drawback to his value as a logical training as well as to his system of Geometry. Many students of Euclid are incapable of distinguishing between a failure in a mathematical proof and in the observance of a conventional rule. Can it be called a good training in logic which obscures the distinction between what is artificial and what is necessary? Do not many mathematicians, trained in Euclid, say that trisecting an angle or squaring a circle is a *Geometrical* impossibility, from this confusion of ideas? And again, Euclid's treatment of parallels distinctly breaks down in Logic. It rests on an axiom which is not axiomatic. It is often said that mathematicians shew more anxiety about their reasoning than about their premises. Whether this be so or not, mathematicians have Euclid's example to plead.

There is moreover a logic besides that of mere reasoning. It is a most valuable discipline in logic to train the mind to scientific order; to teach the relation of general truths to each other, and to the particular cases which fall under them; to make science light its own way by means of a natural arrangement. Now all this cannot be taught by Euclid at all, and might be taught by a well-arranged Geometry.

In short, the logical training to be got from Euclid is very imperfect, and in some respects bad; and yet to this imperfect logical training, such as it is, we confessedly sacrifice our Geometrical teaching: while it is possible to arrange Geometry in a lucid order, and adhering to rigorous methods of demonstration, teach the logic as inseparable from the science.

The very parts of Euclid's Geometry which are most admired by mathematicians, for their ingenuity and completeness, as for example his definition of Proportion in Book v., are open to fatal objections. "Common sense," says De Morgan, "requires that we should satisfy the notion of proportion existing in the mind previously to entering on Geometry as evinced by sameness of relative magnitude, and not invent a new notion for the occasions of Geometry." The reasoning is exquisite and profound; it is too exquisite; it leaves on most men's minds the half-defined impression that all profound reasoning is something far-fetched and artificial, and differing altogether from good clear common sense.

In common then with some of our ablest mathematicians, and with many who are engaged in teaching mathematics, I am of opinion that the time is come for making an effort to supplant Euclid in our schools and universities. Already the fifth book has practically gone; and in consequence the study of the sixth book has become somewhat irrational.

For the improvement of our Geometrical teaching in England two things seem to be wanted. First, that Cambridge, and the Government Examiners, should follow the example of Oxford and London, and examine not in Euclid only, but in the Geometry of specified subjects, according to a programme for each examination. Secondly, that text-books should be written to illustrate what is required. This book is the first part of such a text-book of Elementary Geometry. Probably its method of treating propositions may be considered as sufficiently different from that common in England to justify its publication. At the same time it may be remarked, that the forthcoming second part, which is to embrace the Geometry of the Circle and the applications of proportion to Plane Geometry, ought to bring out in a stronger light than is here possible the superiority of modern to ancient methods of Geometry. The present Part may be used either as introductory to Euclid, or as replacing the first two books. In a few years

I hope that our leading mathematicians will have published, perhaps in concert, one or more text-books of Geometry, not inferior, to say the least, to those of France, and that they will supersede Euclid by the sheer force of superior merit.

In the compilation of this little book, I lay claim to no originality. I have read several French Geometries, and am under some obligations to them. I owe more to the valuable suggestions of my colleague the Rev. C. E. Moberly, who has the spirit without the prejudices of a geometrician. But much of what is most characteristic in the book is due to Dr Temple. It was at his wish that I undertook the work, as he is strongly impressed with the need of it; and his criticisms and his contributions to it have enabled me to rearrange it and improve it in some important respects. And this gives me confidence in publishing it. At the same time we feel that the experience of a few years in teaching with this book will enable me to make improvements in it, without departing widely from the lines here laid down. To Professor Hirst also I have the pleasure to express my thanks for some corrections and remarks.

The distinctive features of this work are intended to be the following. The classification of Theorems according to their subjects; the separation of Theorems and Problems; the use of hypothetical constructions; the adoption of independent proofs where they are possible and simple; the introduction of the terms *locus*, *projection*, *&c.*; the importance given to the notion of direction as the property of a straight line; the intermixing of exercises, classified according to the methods adopted for their solution; the diminution of the number of theorems; the compression of proofs, especially in the later part of the book; the tacit, instead of explicit, reference to axioms; and the treatment of parallels.

J. M. WILSON.

RUGBY,
April 28, 1868.

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INTRODUCTION.

THE notions of *volume*, or *solid content*, *surface*, *line*, are obtained by experience from external objects. The science of *geometry* deals with the properties and construction of *figures*, that is of volumes, surfaces and lines considered independently of the substances of which they are composed. *Plane* geometry deals only with the line and the plane or flat surface; and the *elements* of plane geometry include the properties only of the straight line and circle, and of combinations of straight lines and circles. In the present book we shall treat only of certain properties of straight lines; and we prefix some definitions, and some obvious truths concerning straight lines.

An axiom is a truth so obvious that it needs no demonstration.

Def. 1. A *surface* is the boundary of a solid.

Def. 2. A *line* has position, length and direction, but is not considered as having breadth or thickness. The boundaries of a surface are lines.

Def. 3. *A point* has position, but is not considered as having magnitude. The intersection of two lines, and the extremities of a line are *points.*

Def. 4. *A straight line* is a line which has the same direction at all parts of its length.

Several properties of straight lines are axiomatic; for example, it is obvious

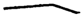
(1) That only one straight line can be drawn to join two given points.

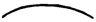
(2) That a straight line marks the shortest distance between any two points in it.

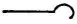
(3) That two straight lines which are not coincident can only have one point in common.

(4) That if two straight lines have two points in common they coincide wholly.

Other lines are called *broken*, *curved*, or *mixed*, as







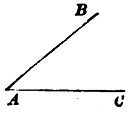
Def. 5. *A plane surface* is one in which, any two points being taken, the straight line which joins them lies wholly in that surface.

Hence the practical test of a plane surface is that the straight edge of a ruler shall lie in contact with it in all positions.

Def. 6. Two straight lines that meet one another have different directions, and the difference of their direction is the *angle* between them.

Thus the direction of AC is different from the direction of AB, and the difference of their direction is the angle

BAC. This angle is also named by a single letter, *A*, at the point where the lines meet, where there can be no ambiguity. *A* is called the *vertex*, *AC*, *AB* the *arms* of the angle, which are said to *form*, or *include*, or *contain* the angle. It is obvious that the angle, or the difference of direction of the lines, does not depend on the length of the lines.



An angle may be conceived as generated by the revolution of a line *AB*, starting from some initial position *AC*, and the angle is the *quantity of turning* required to make *AC* coincide with *AB*.

The explanation of certain terms and signs used in Geometrical reasoning may be introduced here.

A theorem is a geometrical truth which is capable of demonstration by reasoning from certain known truths.

A corollary to a theorem is a geometrical truth easily deducible from that theorem.

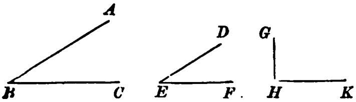
The sign = is used to represent equality. Thus $A = B$ is written to express that *A* is equal to *B*.

The signs $> <$ are used to express 'is greater than,' 'is less than:' thus $C > D$ expresses that *C* is greater than *D*.

The sign + is used to express addition, and – to express subtraction.

The sign $\therefore$ is used for *therefore*, and $\because$ for *because*.

SECTION I. PLANE ANGLES.



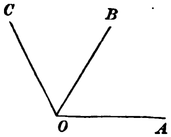
Def. 7. *Angles* are said to be *equal* when they could be placed on one another so that their vertices would coincide in position and their arms in direction.

Thus the angles *B*, *E* are said to be equal, if *E* can be conceived as placed on *B*, *EF* having the same direction as *BC*, and *ED* having the same direction as *BA*.

Conversely, angles which are equal can be conceived as placed on one another so as to coincide.

It is further obvious that angles which are not equal could by no means be so placed as to coincide. Thus the angle *B* cannot be conceived as coinciding with *H*.

Angles are *added* by placing them so as to have a common vertex, and one common arm, their other arms lying on *different* sides of the common arm.



And angles are *subtracted* by placing them in the same manner, but on the *same* side of the common arm.

Thus *AOC* is the *sum* of the angles *AOB*, *BOC*; and *AOB* is the *difference* of the angles *AOC*, *COB*.

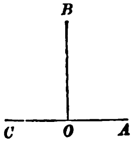
Def. 8. The *bisector* of an angle is the line which divides it into two equal angles.

Def. 9. Angles are said to be *adjacent* when they have a common vertex and one common arm, and their other arms lie on different sides of the common arm.

Def. 10. When one straight line stands upon another straight line so as to make the adjacent angles equal, each of these angles is called *a right angle*, and the line is said to be *perpendicular* to the other, or *at right angles* to the other.

If the adjacent angles are not equal the line is said to be *oblique* to the other.

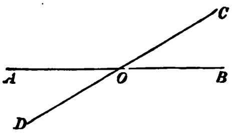
Thus if the angle $AOB =$ the angle BOC, then each of these angles is said to be a right angle, and BO is said to be perpendicular to AC, or at right angles to AC.



Hence the angle between AO and OC, which is the sum of the angles AOB and BOC, may be considered as an angle of two right angles: that is, the quantity of turning by which any line OA may be made to coincide with its continuation is two right angles.

Def. 11. Two angles are said to be *vertically opposite* to one another when they have the same vertex, and the sides of the one are the prolongations of the sides of the other.

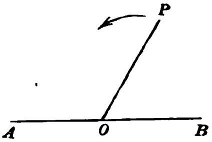
Thus AOD, BOC are vertically opposite angles; and AOC, BOD are also vertically opposite angles.



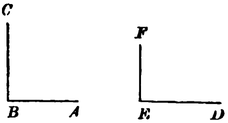
Def. 12. Two angles are said to be *supplementary* to each other when their sum is two right angles.

THEOREM I.

At a given point in a given straight line there can be one perpendicular, and only one, to that line.



For let AB be the given line, O the given point in it. Then, if a line OP drawn from O be conceived to revolve round O in the direction of the arrow, from an initial position coinciding with OB, it is clear that the angle POB will at first be very small, and less than POA, and will continually increase, as the direction of OP alters, and POA will continually diminish, until the angle POB has become greater than the angle POA. Therefore there must be one position, and there can be only one, in which the angle POB is equal to the angle POA.



COR. Hence all right angles are equal. For let B, E be right angles: if they were placed so as to have a common vertex and one common side, as in Def. (7), their other sides must necessarily coincide, for otherwise there would be two perpendiculars drawn to the same straight line from the same point in it.

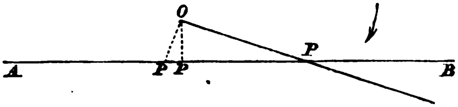
A right angle is therefore a natural standard to which other angles may be referred. It is a sufficient description of the magnitude of an angle to say that it is two-thirds of a right angle; or of the sum of two angles to say that it is equal to the sum of two right angles.

Def. 13. An *obtuse angle* is one which is greater than a right angle.

Def. 14. An *acute angle* is one which is less than a right angle.

THEOREM 2.

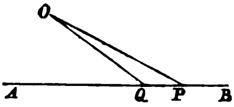
From a given point outside a given straight line of indefinite length one perpendicular, and only one, can be drawn to that straight line.



Let O be the given point, AB the given straight line of indefinite length.

Conceive a line OP revolving round O in the direction indicated by the arrow; then OP will continually be altering its direction relatively to AB; and at first the angle OPB is very large, and is greater than the angle OPA. As the line revolves the angle OPB becomes continually smaller, and OPA becomes continually larger, until ultimately the angle OPA is larger than the angle OPB. Therefore in one position, and in only one, the angle OPB is equal to the angle OPA. That is from the point O one perpendicular, and only one, can be drawn to the straight line AB.

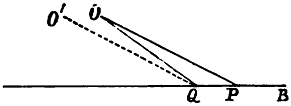
COR. If two straight lines drawn from one point meet a third straight line, they make with it taken in either direction unequal angles, the exterior angle being greater than the interior and opposite.



For from the nature of the reasoning given above it follows that

$$OPB \text{ is} > OQB, \text{ and } OQA > OPA.$$

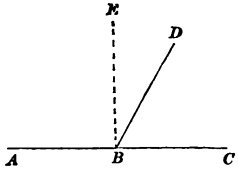
NOTE. It is assumed as self-evident in the proof above given, that the angle OPB *continually* decreases; that is, that the angle OPB in the figure here given is greater than the angle OQB. This fact can be made however to follow from the test of equality and inequality in angles given above, viz., superposition. For if the angle OPB were applied to the angle OQB, the application being conceived to be effected by sliding the angle OPB along the line BQ till P reached Q, it is obvious that the side PO of the angle OPB would fall outside the side OQ of the angle OQB; and therefore the angle OPB is greater than the angle OQB.



THEOREM 3.

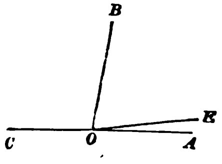
The sum of two adjacent angles, when their exterior arms are in one straight line, is two right angles; and conversely, if two angles whose sum is two right angles are adjacent, their exterior arms shall be in one straight line.

For if the adjacent angles are equal, they are two right angles by definition. And if they are unequal they shall be together equal to two right angles.



For let ABD, DBC be adjacent angles, AB and BC being in one straight line: then the angles CBD and DBA together make up the angle between BC and its continuation BA, that is, they are equal to two right angles.

Conversely: let BOA, BOC be together equal to two right angles, and be placed so as to have a common vertex O and one common arm OB. Then their other arms CO and OA shall be in one straight line.

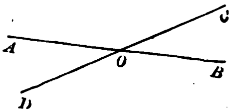


For if BOC, BOA were equal to two right angles, and yet OA not in the same straight line with CO, conceive OE in the same straight line with OC.

Then BOE and BOC would be together equal to two right angles, and would therefore be equal to BOA and BOC, which is impossible. Therefore OA must be in the same straight line with OC.

Cor. 1. Of two adjacent angles, when their exterior arms are in one straight line, each is supplementary to the other.

Cor. 2. Vertically opposite angles are equal.



For let AOD, COB be vertically opposite angles.

Then, since each of them is the supplement of AOC, they must be equal to one another;

that is, since $AOC + COB =$ two right angles,

and $AOC + AOD =$ two right angles;

$\therefore$ $AOC + COB = AOC + AOD$;

and $\therefore$ $AOD = COB$.

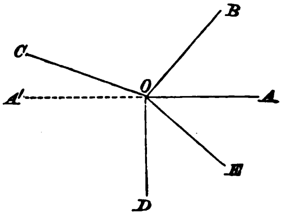
It must be observed that a straight line marks two opposite directions; thus the direction OC is opposite to the direction CO.

Hence the angle COB is the difference of the directions of DC and AB, and AOD is the difference of the directions of CD and BA; that is, of the two former lines taken both in the reverse direction. And it is clear that the quantity of turning required to bring OB into coincidence with OC, would also bring OA into coincidence with OD, that is, the angle $BOC =$ the angle AOD.

Cor. 3. The sum of all the angles taken consecutively, made by any number of lines which meet at one point, is four right angles.

Let OA, OB, OC, OD, OE be straight lines meeting in one point, then the sum of the angles AOB, BOC, COD, DOE, EOA will be four right angles.

For produce one of the lines AO to A'.



Then $AOB + BOC$ are less than two right angles by COA', and $COD + DOE + EOA$ are greater than two right angles by the same angle COA'. Therefore

$$AOB + BOC + COD + DOE + EOA = \text{four right angles.}$$

Hence if the line OB be conceived to revolve round O, starting from an initial position OA, when it has performed one complete revolution, and has returned to the same position, it will have described four right angles.

Exercises on Angles.

1. If two straight lines intersect at a point, and one of the four angles at that point is a right angle, prove that the others also are right angles.

2. If five lines meet at a point, and make equal angles round that point, each of the angles is four fifths of a right angle.

3. Of two supplementary angles the greater is double of the less, find what fraction the less is of four right angles.

4. From a given point in a given straight line two straight lines, and only two, can be drawn on the same side of it so as to make a given angle with that line.

5. From a given point outside a given straight line two straight lines, and only two, can be drawn so as to make a given angle with that line.

6. If four straight lines OA, OB, OC, OD meet at a point O, and $AOB = DOC$ and $AOD = COB$, prove that AOC, BOD are straight lines.

7. The bisectors of the four angles which one straight line makes with another are two straight lines at right angles to one another.

SECTION II. PARALLELS.

The previous Section has treated of lines that have *different directions*, and therefore include an angle. In the present section we shall discuss some properties of lines that have the *same direction.*

Def. 15. Straight lines which have the *same direction*, but are not parts of the same straight line, are called *parallel lines.*

It is obvious from the definition:

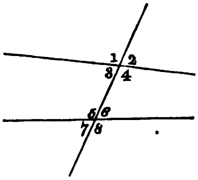
(1) That pairs of parallel lines are in the same plane.

(2) That parallel lines would never meet however far they were produced; since if they met they would have different directions.

(3) That straight lines which are parallel to the same straight line are parallel to one another.

For straight lines which are parallel to the same straight line have the same direction as that line, and therefore have the same direction as one another. That is, they are parallel.

Def. 16. When a straight line intersects two other straight lines, it makes with them eight angles which have received special names.



In the figure, 1, 2, 7, 8 are called *exterior* angles

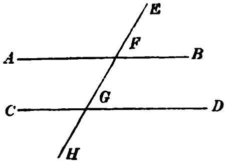
and 3, 4, 5, 6 are called *interior* angles.

Again, 1 and 5 are said to be *corresponding* angles; so also are 2 and 6, 7 and 3, 8 and 4; and 3 and 6 are called *alternate* angles, so also are 4 and 5.

Corresponding angles then are those which the intersecting line, *taken in one direction*, makes with the other two lines on the same side of it.

THEOREM 4.

If two parallel lines are cut by a third line, the corresponding angles will be equal to one another.



Let *AB* and *CD* be parallel lines, intersected by a third line *EFGH*. Then will the corresponding angles be equal.

For since *AB* and *CD* are parallel, that is, since they have the same direction, therefore the difference between their direction and that of *EH* is the same. But the angle *BFG* is the difference of direction of *AB* and *EH*, and the angle *DGH* is the difference of direction of *CD* and *EH*; therefore the angle *BFG* = the angle *DGH*.

And in the same manner it may be shewn that any angle in the figure is equal to its corresponding angle.

COR. 1. It follows that the alternate angles are equal. For *DGH*=*FGC* which is vertically opposite to it; and therefore also *BFG*=*FGC*.

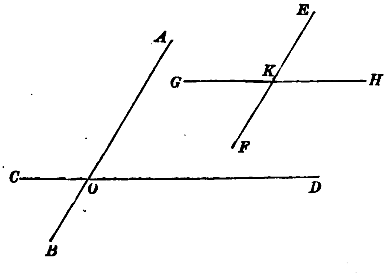
Cor. 2. Of the two interior angles on the same side of the intersecting line, each is supplementary to the other.

For *FGD* is supplementary to *DGH* (Th. 3. Cor. 1), and therefore also to *BFG*.

Theorem 5.

If two lines are respectively parallel to two other lines, the angles made by the first pair will be equal or supplementary to the angles made by the second pair; equal, if both are taken in the same, or both in the opposite direction; supplementary if one is taken in the same and one in the opposite direction.

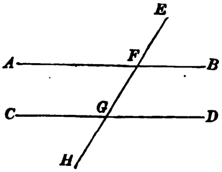
For if *AB* and *CD* have respectively the same directions as *EF* and *GH*, the difference of the directions of *AB* and *CD* is the same as the difference of the direction of *EF* and *GH*. That is, the angle *BOD* is equal to *FKH*, and is therefore also equal to *GKE*; and so also *AOD* or *BOC* is equal to *EKH* or *GKF*. And the angle *EKG*, which is supplementary to *EKH*, is therefore also supplementary to *AOD*.



THEOREM 6.

If two straight lines make with a third straight line intersecting them equal corresponding angles, these two lines will be parallel.

Let *AB* and *CD* make with *EFGH* equal corresponding angles *EFB*, *FGD*; then shall *AB* and *CD* be parallel.



For since *AB* and *CD* make equal angles with the same line *HE* on the same side of it, they differ equally in direction from *HE* on the same side of it, and therefore they have the same direction, that is, they are parallel.

COR. 1. If the alternate angles are equal, the lines are parallel.

For if $AFG = FGD$, then also $EFB = FGD$, since $AFG = EFB$.

COR. 2. If the two interior angles on the same side of the intersecting line are together equal to two right angles, the lines will be parallel.

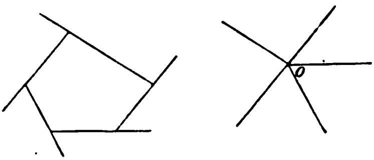
For if *BFG* is supplementary to *FGD*, and is also supplementary to *EFB*, then $EFB = FGD$, and the lines are parallel.

COR. 3. If the two interior angles on the same side of the intersecting line were together less than two right angles, the lines would meet on that side of the intersecting line.

Def. 17. A figure enclosed by any number of straight lines is called a *polygon.* It is called *convex* when no one of its sides when produced intersects the polygon. It is called *regular* when all its sides and angles are equal.

THEOREM 7.

The exterior angles of any convex polygon, made by producing the sides in succession, are together equal to four right angles.



Conceive lines drawn through any point O parallel to the sides of the polygon, in the direction in which the sides are produced.

Then (by Theorem 5) the angles taken consecutively round O are equal to the exterior angles of the polygon. But the angles at O are together equal to four right angles. Therefore the exterior angles of the polygon are together equal to four right angles.

Cor. Hence it may be shewn that all the interior angles of any polygon are less than twice as many right angles as the figure has sides by four right angles.

For each interior angle with its adjacent exterior angle = two right angles;

∴ all the interior angles + all the exterior angles = twice as *many right* angles as the figure has sides;

But all the exterior angles = four right angles;

∴ all the interior angles + four right angles = twice as many right angles as the figure has sides;

and therefore all the interior angles are less than twice as many right angles as the figure has sides by four right angles.

Hence the magnitude of the angle of any *equiangular* polygon can be found.

Let it be required, for example, to find the magnitude of the angle of a regular hexagon. Since the six interior angles are less than twelve right angles by four right angles, they are together equal to eight right angles; and therefore each of them is eight-sixths of a right angle, or is $1\frac{1}{3}$ of a right angle.

EXERCISES ON PARALLELS.

1. Shew that the angles of an equilateral triangle are each equal to two-thirds of a right angle.

2. Find the magnitude of the angle of a regular octagon.

3. Shew that three hexagons can be placed so as to have one common angular point, and fill up the space round that point.

4. Shew that two octagons and one square have the same property; and hence shew that octagons and squares having sides of the same length can be arranged so as to cover any space.

5. How many equilateral triangles can be placed so as to have one common angular point?

6. Compare the angle of a pentagon with the angle of a decagon.

7. If two lines intersecting in *A* are respectively perpendicular to two lines intersecting in *B*, prove that the angles at *A* are respectively equal or supplementary to the angles at *B*.

SECTION III. PLANE TRIANGLES.

Def. 18. *A triangle* is the figure contained by three straight lines, which are called its *sides.*

Def. 19. A triangle is called *isosceles* when two of its sides are equal.

Def. 20. A triangle is called *equilateral* when all its sides are equal, and *equiangular* when all its angles are equal.

Def. 21. It is said to be *right-angled* or *obtuse-angled* when one of the angles is a right angle or an obtuse angle, and *acute-angled* when all its angles are acute angles.

Def. 22. A triangle is sometimes regarded as standing on one of its sides, which is then called its *base;* and the angle opposite that side is called the *vertex.* When two of the sides of a triangle have been mentioned, the remaining side is often spoken of as the *base.*

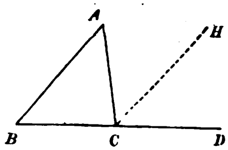
Def. 23. The term *hypothenuse* is used to describe the side of a right-angled triangle which is opposite to the right angle.

Def. 24. Triangles are said to be *equal in all respects* when they have the three sides and the three angles of the one equal to the three sides and the three angles respectively of the other, and the area of one equal to the area of the other.

Def. 25. The *area* of a figure is the space enclosed by the sides of the figure.

THEOREM 8.

If one side of a triangle is produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of any triangle are together equal to two right angles.



Let the side BC of the triangle ABC be produced to D: then shall the angle $ACD =$ the sum of the angles ABC, BAC. And the three angles ABC, BCA, CAB shall be together equal to two right angles.

For if through C a line CH were drawn parallel to BA
the angle $HCD =$ the corresponding angle ABC,
and angle $ACH =$ the alternate angle BAC;
$\therefore$ the whole angle $ACD =$ the two angles $ABC + BAC$.

And if ACB be added to these,
the two angles $ACD + ACB =$ the three angles $ABC + BCA + CAD$.

But $ACD + ACB =$ two right angles;
therefore $ABC + BCA + CAD =$ two right angles.

Remark. This result is obviously a particular case of the preceding Theorem.

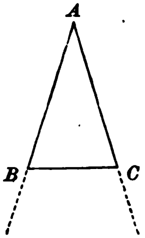
COR. 1. It follows that no triangle can have more than one right angle or obtuse angle.

COR. 2. In a right-angled or obtuse-angled triangle the right or obtuse angle is the greatest angle.

COR. 3. In any right-angled triangle the two acute angles together make up one right angle.

THEOREM 9.

An isosceles triangle has the angles at its base equal.



Let ABC be an isosceles triangle, having $AB = AC$.

Then shall the angle $B =$ the angle C.

For conceive the triangle ABC taken up and put down in its former position, with its sides reversed. That is, let A be put down where it was before, and AC where AB was; then AB would be where AC was, since the angle at A is unchanged, and B would fall where C was, and C where B was, because $AB = AC$; and BC would fall as it was before, but in a reversed position; that is, the angle B would fall on the angle C, and therefore the angle $B =$ the angle C.

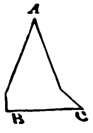
COR. 1. The angles on the other side of the base, made by producing the equal sides, are equal.

For they are respectively supplementary to the angles at the base.

COR. 2. An equilateral triangle is equiangular.

For every pair of the angles is equal.

THEOREM 10. Conversely.



A triangle which has the angles at its base equal is isosceles.

Let the triangle ABC have the angles B and C equal.

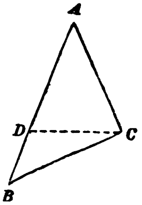
Then shall $AB = AC$.

For conceive the triangle taken up and put down in its former position, but with its sides reversed; that is, let the point C be placed where B was, and B where C was. Then CA would have the same direction that BA had before, and BA would have the same direction that CA had before, because the angle $B =$ the angle C; therefore A must lie where it did before, and therefore BA would fall on CA, and therefore $BA = CA$.

Cor. An equiangular triangle is equilateral.

THEOREM 11. Oppositely to 9.

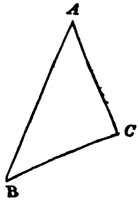
A triangle which has two of its sides unequal shall have the angles at its base unequal, the greater angle being opposite to the greater side.



Let AB be $>$ AC. Then shall the angle ACB be $>$ the angle ABC. Cut off from AB a part $AD = AC$, and join DC. Then the angle $ADC =$ the angle ACD, since the triangle ADC is isosceles. But the angle ACB is $>$ ACD, and ABC is $<$ ADC (Th. 2. Cor.); much more then is ACB greater than ABC.

THEOREM 12. Conversely to 11.

A triangle which has two of its angles unequal shall have the sides opposite them unequal, the greater side being opposite to the greater angle.



Let the angle ACB be greater than the angle ABC, then shall AB be greater than AC.

For AB is not equal to AC, for then (by Th. 9) the angle ABC would be equal to the angle ACB.

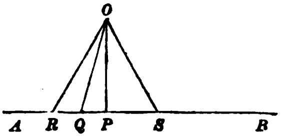
Nor is AB less than AC, for then the angle ACB would be less than ABC by (11);

Therefore AB is greater than AC.

THEOREM 13.

Of all the straight lines that can be drawn from a given point to meet a given straight line, the shortest is the perpendicular; and of the others, that which is further from the perpendicular is greater than one which is nearer; and one, and only one, oblique can be drawn equal to any given oblique, and it will be on the other side of the perpendicular and equally inclined to it.

Let O be the given point, AB the given straight line. Then (by Theorem 2) there can be one line OP perpendicular to AB.



Let OQ be any other line from O meeting AB in Q; OP shall be less than OQ.

For since OPQ is a right angle, OQP is an acute angle (Th. 8, Cor. 2); that is $OPQ > OQP$,

and therefore $OQ > OP$ by Theorem 12.

Hence OP is less than *any* other line OQ, and therefore is the least of all the lines drawn from O to AB.

Next let OQ, OR be two obliques of which OR is the further from the perpendicular; then OR shall be greater than OQ.

For since OPQ is a right angle, OQR the exterior angle is an obtuse angle (Th. 2, Cor.) and ORQ is an acute angle; therefore OQR is $>$ ORQ,

and therefore OR is $>$ OQ (Theorem 12).

Lastly, there can be drawn one and only one oblique equal to a given oblique, which will be equally inclined to the perpendicular and on the opposite side of it.

For if $PS = PR$, and the figure be conceived as folded on OP, it is clear that since the angles at P are right angles, PS would fall on PR, and S on R, and therefore OS on OR. Therefore $OS = OR$. Moreover the angle $POS =$ the angle POR, that is OS and OR are equally inclined to the perpendicular OP on opposite sides of it.

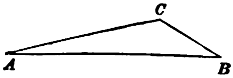
Hence too it is evident that not more than two equal obliques can be drawn, for of the two on one side of the perpendicular one must be more remote from it than the other, and therefore must be greater than the other.

COR. Of two obliques, whether on the same or opposite sides of the perpendicular, the longer is more remote from the perpendicular than the shorter.

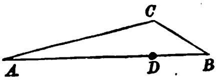
THEOREM 14.

Any two sides of a triangle are together greater than the third side.

For in the triangle ACB, AB is the straight line joining A and B, and is therefore shorter than the broken line ACB, by one of the axioms respecting straight lines.



COR. It follows that the difference of any two sides of a triangle is less than the third side; for example, that the *difference of AB* and AC is less than CB.

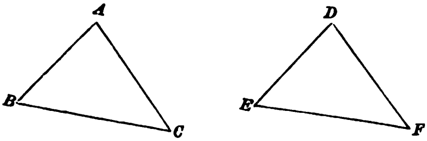


For from *AB* cut off $AD = AC$.

Then *DB* is the difference between *AB* and *AC*. Since therefore $AC + CB > AD + DB$, and $AC = AD$, it follows that *CB* is greater than *DB*; that is, *DB* is less than *CB*.

THEOREM 15.

If two angles of one triangle are equal respectively to two angles of another triangle, then shall the third angles of the triangles be equal.



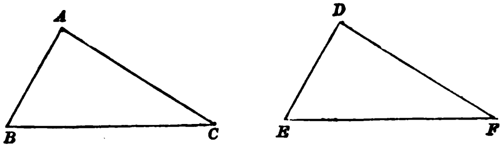
Let the two angles *B* and *C* of the triangle *ABC* be respectively equal to *E* and *F* of the triangle *DEF*. Then shall the angle *A* be equal to the angle *D*.

For *A*, *B*, and *C* together make up two right angles, and so also do *D*, *E*, and *F*. Therefore, since *B* and *C* are equal to *E* and *F*, it follows that *A* must be equal to *D*.

THEOREM 16.

If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, these triangles will be equal in all respects.

If two angles of one triangle are respectively equal to two angles of another triangle, it follows (by Theorem 15) that the third angles of the two triangles are respectively equal.



Let *ABC*, *DEF* be the two triangles, in which one side *BC* is equal to one side *EF*, and the angles *A*, *B*, *C* are respectively equal to their corresponding angles *E*, *D*, *F*.

Then shall the triangles be equal in all respects.

For if the triangle *ABC* be conceived as placed on the triangle *DEF*, so that the side *BC* coincides with the equal side *EF*, then *BA* will fall on *ED*, since the angle *B* = the angle *E*; and *CA* will fall on *FD*, since the angle *C* = the angle *F*.

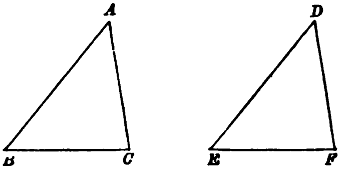
Therefore *A* will fall on *D*, and the triangles will coincide, and are therefore equal in all respects: that is, $AB = DE$, $AC = DF$, and the area of the triangle $ABC =$ the area of the triangle *DEF*.

THEOREM 17.

If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the triangles will be equal in all respects.

Let the two sides *BA*, *AC* of the triangle *BAC* be *respectively* equal to the two sides ED, DF of the triangle

EDF, and let them contain the angle BAC = the angle EDF; then will the triangles be equal in all respects.



For the angle at A can be conceived as placed on the angle D, since these angles are equal, AB lying on DE, and AC on DF; and B and C would then coincide respectively with E and F, since $AB = DE$ and $AC = DF$.

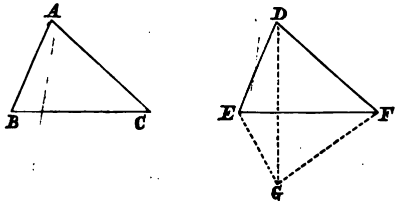
Hence BC would fall on EF, since these lines would have two points in common; and therefore BC must be equal to EF, and the angles B and C respectively coincide with and are equal to the angles E and F; and the area of the triangle ABC is equal to the area of the triangle DEF.

THEOREM 18.

If the three sides of one triangle are respectively equal to the three sides of another, these triangles will be equal in all respects.

Let the three sides of the triangle ABC be respectively equal to the three sides of the triangle DEF; then shall the triangles be equal in all respects.

For conceive the triangle ABC placed so that the base BC coincides with the equal base EF, but so that the vertex *A* lies on the opposite side of *EF* from *D*, as at *G*; and join *DG*.



Then *EDG* and *FDG* are, by the hypothesis, isosceles triangles; and therefore the angles *EDG*, *FDG* are respectively equal to the angles *EGD*, *FGD*: and therefore the whole angle *EDF* is equal to the whole angle *EGF*, that is, to *BAC*: and hence the triangle *BAC* could be placed so as to coincide with the triangle *EDF*, and is therefore equal to it in all respects.

Remark. The student should examine for himself the slight modification that must be made in this proof when the shape of the given triangles is such that the line *DG* passes through an extremity of *EF*, or intersects *EF* produced.

NOTE.—By *corresponding* angles in any two triangles are meant those which are opposite to equal sides; and by *corresponding* sides are meant those which are opposite to equal angles.

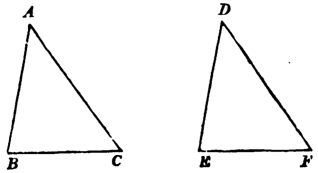
THEOREM 19.

If two triangles have two sides of the one equal to two sides of the other, and the angle opposite that which is not the less of the two sides of the one equal to the corresponding angle of the other, the triangles shall be equal in all respects.

Let ABC, DEF be the two triangles, having the sides BA, AC equal to the sides ED, DF respectively, of which AC is not less than AB, and having also the angle $B =$ the angle E.

Then shall the triangles be equal in all respects.

For if $AC = AB$, and $DE = DF$, the angles B, C, E, F are equal, and the remaining angle $A =$ the remaining angle D, and the triangles are equal in all respects by Theorems (16) or (17).



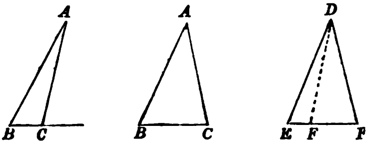
If AC is $> AB$, then since $AB = DE$ it could be placed so as to coincide with it; and since the angle $B =$ the angle E, the line BC would coincide in direction with EF, and the point C would fall somewhere on EF, or on EF produced through F.

It remains to see the effect of the only remaining condition that $AC = DF$.

Now the line AC has one extremity A on D, and the other extremity C somewhere on EF, or on EF produced through F: and $AC = DF$. But there is no oblique equal to DF, and not coinciding with it, that can be drawn from D to any point in EF; for since DF is $> DE$ it is more remote from the perpendicular; and therefore the oblique equal to DF would lie on the other side of DE, and have its extremity in EF produced through E.

Hence AC would coincide with DF, and the triangles are therefore equal in all respects.

COR. 1. If AC the side opposite the given angle were less than AB there would be two triangles ABC, as in the figure, both of which have the two sides $BA, AC = ED, DF$, and the angle B equal to the corresponding angle E; and the proof given above is inapplicable.



This is called *the ambiguous case.*

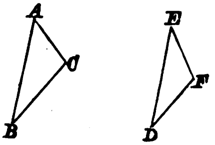
COR. 2. If the given angle is a right angle, the side opposite to it must be greater than the side adjacent; by Th. 12. Hence if two right angled triangles have the hypothenuse and one side of the one equal respectively to the hypothenuse and one side of the other, the triangles are equal in all respects.

COR. 3. A similar property is obviously true of two obtuse-angled triangles.

THEOREM 20.

If two sides of one triangle are respectively equal to two sides of another, but the included angles are unequal, the bases shall be unequal, that base being the greater which is opposite the greater angle.

Let the two sides BA, AC of the triangle BAC be respectively equal to the two sides *DE*, *EF* of the triangle



DEF, but let the angle *BAC* be greater than the angle *DEF*: then shall the base *BC* be greater than the base *DF*.

For if the vertex *E* were placed on *A*, and one side *ED* placed on the equal side *AB*, *D* would fall on *B*: but *EF* would not fall on *AC*, but would lie nearer to *AB* because the angle *DEF* is less than the angle *BAC*. The point *F* might fall outside the triangle *BAC*, or on the line *BC*, or inside the triangle *BAC*, according to the

Fig. 1.

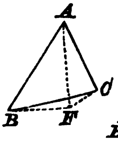


Fig. 2.

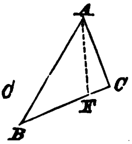
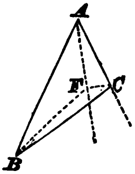


Fig. 3.



shapes of the triangles, as in figures 1, 2, 3; and in each case it may be shewn that *BC* is greater than *BF*, that is, than *DF*.

In figure 2 no demonstration is required; but in figures 1 and 3 let *FC* be joined, and in figure 3 let *AF*, *AC* be produced.

Then since *AFC* is an isosceles triangle, for $AF = AC$, it follows that in fig. 1, the angles at its base, and, in fig. 3, the angles on the other side of the base, are equal. But it is obvious that *BFC* is greater than one of these angles, and *BCF* is less than the other. Therefore BFC is > BCF, and therefore BC is $>$ BF by Theorem 12, and therefore BC is $>$ DF.

COR. The converse of this theorem may be proved as in Theorem 12.

EXERCISES ON TRIANGLES.

The theorems respecting triangles, which follow from the elementary properties of triangles proved above, are of two kinds; which announce respectively the equality, or the inequality of lines and angles.

Theorems of equality depend on Theorems 8, 9, 10, 15, 16, 17, 18, 19, and theorems of inequality on Theorems 11, 12, 13, 14, 20, given above.

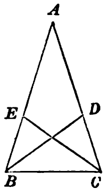
We shall give a few examples of these theorems with demonstrations, and add some exercises which may be left to the student's ingenuity to prove. They require an application of these general theorems to the special data of the theorem proposed.

The general method to be adopted in the solution of theorems of equality is the following. Examine fully the statement of the question; see what is included among the data: what lines and angles are given equal. Then see what is required to be proved, what lines or angles have to be proved to be equal. Examine the triangles of which they form corresponding parts, and see whether the data are sufficient to prove *these* triangles equal. If the data are sufficient, the solution is effected by comparing the triangles, and shewing the required equality of the lines and angles; if not, the data must be used to establish results, which in their turn can be used to establish the conclusion required.

Theorems of inequality are proved by tracing the consequences of the inequalities that are given or implied in the statement of the question, according to the theorems already proved, until the required result is arrived at.

Theorems of Equality.

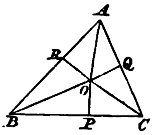
1. The lines which bisect the angles at the base of an isosceles triangle, and meet the opposite sides, are equal.



Let ABC be an isosceles triangle having $AB=AC$, and let BD, CE bisect the angles at the base, then shall $BD=CE$.

For since the angles ECB and DBC are equal, being halves of the angles ACB, ABC, which are equal (since the triangle is isosceles), we see that in the triangles EBC, DCB, the two angles EBC, ECB and the side BC of the one, are respectively equal to the two angles DCB, DBC and the corresponding side of the other. Therefore (Theorem 16) the triangles are equal in all respects, and therefore $BD=CE$.

2. The bisectors of the three angles of a triangle meet in one point.



Let the bisectors of the angles ABC, ACB be BO, CO, meeting in O; then the theorem will be proved if we can shew that AO is the bisector of the angle BAC.

Let perpendiculars OP, OQ, OR be drawn to the three sides BC, CA, AB. Then the triangles OQC, OPC have two angles OQC, OCQ and the side OC of the one equal respectively to the two angles OPC, OCP and the corresponding side OC of the other.

Therefore $OQ = OP$ by Theorem 16.

Similarly from the triangles OPB, ORB, it follows that $OP = OR$, therefore $OR = OQ$;

And therefore the right-angled triangles OQA, ORA have the hypothenuse and one side of the one equal to the hypothenuse and one side of the other, and are therefore equal in all respects by Theorem 19, Cor. 2.

Therefore the angle $OAQ =$ the angle OAR, that is, OA is the bisector of the angle BAC.

EXERCISES ON TRIANGLES.

Theorems of Equality.

1. The line drawn to bisect the vertical angle of an isosceles triangle also bisects the base, and is perpendicular to it.

2. The line drawn from the vertex of an isosceles triangle to bisect the base, cuts it at right angles, and bisects the vertical angle.

3. The perpendicular let fall from the vertex of an isosceles triangle to the base, bisects the base and the vertical angle.

4. The lines joining the middle points of the sides of an isosceles triangle to the opposite extremities of the base are equal to one another.

5. The perpendiculars let fall from the extremities of the base of an isosceles triangle upon the opposite sides are equal, and make equal angles with the base.

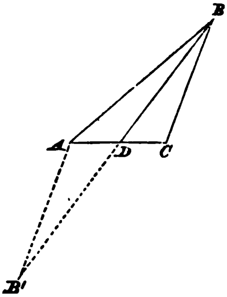
6. If two exterior angles of a triangle be bisected by *straight lines* which meet in O, prove that the perpendicu-

lars from O on the sides or sides produced of the triangle are equal to one another.

7. If two isosceles triangles ABC, ADC are upon the same base AC, and BD be joined, prove that BD bisects the vertical angles of both the triangles, and is perpendicular to the base.

Theorems of Inequality.

1. The line that joins the vertex to the middle point of the base of a triangle is less than half the sum of the two sides.



Let D be the middle point of AC, then is BD less than half the sum of AB, BC.

Produce BD to B', making $DB' = DB$. Join AB'.

Then since the two triangles BDC, ADB' have two sides BD, DC and the included angle BDC of the one respectively equal to the two sides $B'D$, DA and the included angle $B'DA$ of the other, therefore (theorem 17) the base $BC =$ the base AB';

but $$B'A + AB > B'B,$$

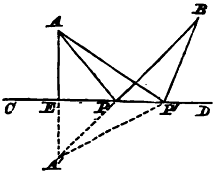
$$\therefore\ AB + BC > B'B, \text{ which is twice } BD,$$

that is, BD is less than half the sum of AB and BC.

2. The sum of the distances of any point in a straight line from two other points on the same side of the line will be least, when the distances make equal angles with the straight line.

Let P be a point in the straight line CD, A, B the other points.

Then $AP+PB$ will be least when AP, PB make equal angles with CD, that is, when $APE=BPD$.



For let the angle APE in the figure be equal to the angle BPD.

Draw AE perpendicular to CD, and produce BP to meet the perpendicular AE produced in A'.

Then the triangles AEP, $A'EP$ will have the right angles at E equal, the angle $APE=$ the angle $A'PE$, since each of them is equal to the angle BPD, and the side EP common; therefore $AP=A'P$, and $AE=A'E$, by Th. 16.

Similarly if P' is any other position of P, $AP'=A'P'$, from the triangles AEP', $A'EP'$, by Th. 17. But $A'P'+BP'>A'B$; that is, $AP'+BP'>AP+BP$; therefore $AP+BP<$ any other pair of distances, and is therefore the least possible when AP, PB make equal angles with CD.

EXERCISES ON TRIANGLES.

Theorems of Inequality.

1. Any one side of a four-sided figure is less than the sum of the other three sides.

2. The sum of the lines which join the opposite angles of any four-sided figure are together greater than the sum of either pair of opposite sides of the figure.

Note. A four-sided figure is called a quadrilateral figure, and the lines that join its opposite angles are called its diagonals.

3. The sum of the diagonals of a quadrilateral figure *is less* than the sum of the four lines which can be drawn

to the angles from any other point than the intersection of the diagonals.

4. O is any point within the triangle ABC; prove that $OA + OB + OC$ are less than the sum, and greater than half the sum of $AB + BC + CA$.

5. Prove that the sum of the four sides of a quadrilateral figure is greater than the sum and less than twice the sum of the diagonals.

6. If ABC is a triangle in which AB is greater than AC, and D is the middle point of BC, and AD is joined, prove that the angle ADB is an obtuse angle.

SECTION IV. PROBLEMS OF CONSTRUCTION.

In the Science of Geometry there are not only theorems to be proved, but constructions to be effected, which are called *problems*. Geometers have always imposed certain limitations on themselves with respect to the instruments which might be used in these constructions. There is no reason why any convenient instrument used in drawing should not be supposed to be used in the Science of Geometry, such as squares, parallel rulers, elliptic compasses; but the ruler and compasses suffice for nearly all the simpler constructions, and those which cannot be effected by their means are considered as not forming a part of Elementary Geometry. There are some problems, that seem at first sight not very difficult, that cannot be solved by the use of these instruments. We can, for example, bisect an angle; but we cannot, in general, trisect it, that is, divide it into three equal parts, by any combination of ruler and compasses.

The solution of a problem in Elementary Geometry as above defined consists:

(1) in indicating how the ruler and compasses are to be used in effecting the construction required;

(2) in proving that the construction so given is correct;

(3) in discussing the limitations, which sometimes exist, *within which* alone the solution is possible.

We shall give several examples of such problems, and then discuss the principles of the methods we have used. It will be observed that in the figures we make the given lines thick, the resulting line or lines thin, and the lines used only in the construction or proof dotted.

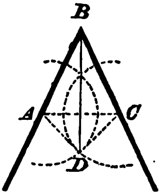
Def. 26. *A circle* is a plane figure contained by a line called the *circumference* such that all the points in that line are equally distant from a certain point which is called the *centre* of the circle. The distance of any point on the circumference from the centre is called the *radius* of the circle.

PROBLEM I.

To bisect a given angle, that is, to divide it into two equal parts.

Construction. Let ABC be the given angle.

Take any equal lengths BA, BC, and join AC. With centre A and any radius greater than half AC describe a circle, and with centre C and the same radius describe another circle intersecting the former circle in D. Join AD, CD, and BD; then BD shall bisect the angle ABC.



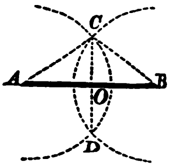
Proof. For the triangles ABD, CBD have obviously the three sides of the one equal (by the construction) to the three sides of the other, and therefore (by Theorem 18) the corresponding angles ABD, CBD are equal.

PROBLEM 2.

To bisect a given straight line.

Let AB be the given straight line.

Construction. With centre A and any radius greater than half AB describe a circle, and with centre B and the same radius describe a circle intersecting the former in two points C and D. Join CD cutting AB in O. Then O will be the point of bisection.



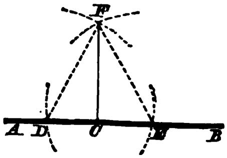
Proof. For by the last Problem CD bisects the angle ACB; and then in the triangles ACO, BCO we have $AC = BC$, CO common, and the included angles ACO, BCO equal: and therefore by Theorem 17, the base $AO =$ the base BO: that is O is the point of bisection of AB.

PROBLEM 3.

To draw a perpendicular to a given straight line from a given point in it.

Let AB be the given line, C the given point in it.

Construction. With centre C and any radius describe a circle to cut the straight line in two points D, E, so that $CD = CE$. With centre D and any radius greater than DC describe a circle, and with centre E and the same radius describe a circle, cutting the former in F. Join FC.



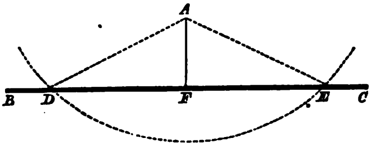
Then FC is perpendicular to AB.

Proof. For the triangles *DCF*, *ECF* have by the construction the three sides of the one equal respectively to the three sides of the other, and therefore the corresponding angles *DCF*, *ECF* are equal to one another, and therefore they are right angles by the Definition.

Note.—This construction is usually effected in practice by means of the square.

PROBLEM 4.

To draw a perpendicular to a given straight line from a given point without it.



Let *BC* be the given straight line, *A* the given point.

Construction. With centre *A* describe a circle with any sufficient radius to cut *BC* in two points *D*, *E*. Bisect *DE* in *F*. Join *AF*.

Then *AF* shall be perpendicular to *BC*.

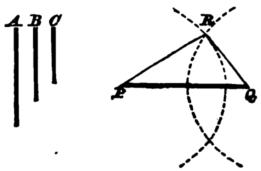
Proof. For if *AD*, *AE* be joined, it is clear that the triangles *AFD*, *AFE* have the three sides of the one respectively equal to the three sides of the other; therefore the angle *AFD* = the corresponding angle *AFE*, and therefore *AF* is perpendicular to *BC*.

Note.—This construction also is usually effected in practice by means of the square.

A large class of problems consists of those in which a triangle has to be constructed under certain conditions. We shall take the most important cases, which are also the simplest.

PROBLEM 5.

To construct a triangle, having given the lengths of the three sides.



Let the three given lengths be the lines A, B, C.

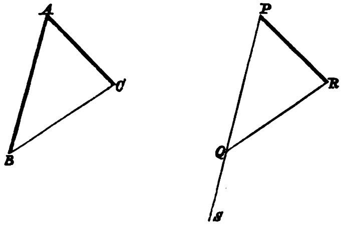
Construction. Draw a line PQ equal to one of them A. With centre P and radius equal to B describe a circle; and with centre Q and radius equal to C describe a circle. Let these circles intersect in R. Join RP, RQ.

RPQ is the triangle required.

Proof. For its three sides are by the construction equal to the three given lines.

Limitation.—It is necessary that any two of the lines A, B, C should be together greater than the third. For if B and C were together less than A, the circles in the figure would obviously not meet: and if they were together equal to A, the point R would be on PQ, and the triangle would become a straight line. Similarly if B were greater than $A + C$ or C greater than $A + B$, the circles would not intersect. This limitation might be anticipated from the theorem before proved, that any two sides of a triangle are together greater than the third side.

COR. *Hence it is possible with the ruler and compasses* alone to effect what is performed in practice generally by means of the sector or protractor, viz. *to make an angle equal to a given angle.*



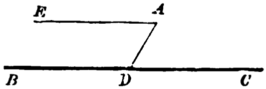
Let BAC be the given angle.

Join any two points B, C in its sides. Construct a triangle PQR having its three sides PQ, QR, RP respectively equal to AB, BC, CA.

Then (by Theorem 14) the angle P is equal to the angle A.

Cor. 1. It is obvious that by this means an angle equal to a given angle may be made at any point, and such that one of the lines containing it may be any given line through the point. Thus if P were the given point, PS the given line, PQ must be taken equal to AB, and the rest of the construction is the same as before.

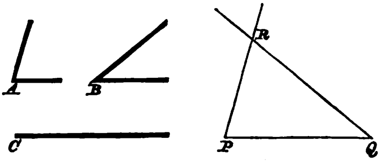
Cor. 2. Hence through any point a straight line can be drawn parallel to a given straight line.



Let A be the given point, BC the given line. Draw any line *AD* to meet *BC*, and make the angle *DAE* equal to the alternate angle *ADC*. Then by Theorem 6, *AE* is parallel to *BC*.

PROBLEM 6.

To construct a triangle, having given two angles and a side adjacent to both.



Let *A*, *B* be the two angles, *C* the given side.

Take a line $PQ = C$. At the points *P*, *Q* make angles equal respectively to *A* and *B*. Let the lines which contain these angles meet in *R*.

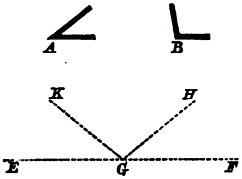
Then *RPQ* is the triangle required.

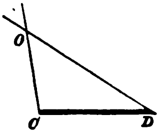
Limitation.—The two given angles must be together less than two right angles, or the lines *PQ*, *QR* would not meet. This follows also from the theorem that the three interior angles of a triangle are together equal to two right angles.

PROBLEM 7.

Having given two angles and a side opposite to one of them, to construct the triangle.

Let *A* and *B* be the given angles, *CD* the given side *which is to* be opposite to *A*.





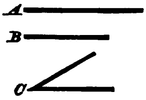
Construction. Draw an indefinite straight line EF. At any point G in it make the angles $FGH=A$, and $HGK=B$, then KGE will equal the third angle of the triangle, since the sum of the three angles of a triangle is equal to two right angles. At C and D make angles equal to HGK and KGE, and let their sides meet in O; then OCD is the triangle required.

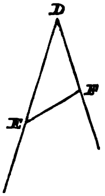
Proof. For the angle O is the supplement of the angles $OCD+ODC$, and must therefore be equal to HGF, that is to A.

Limitation.—As before the two given angles must be together less than two right angles.

PROBLEM 8.

To construct a triangle, having given two sides and the angle between them.





Let A, B be the given sides, C the given angle.

Construction. Draw an angle D equal to the given angle, and take DE, DF equal to A and B. Join EF.

This problem needs no proof.

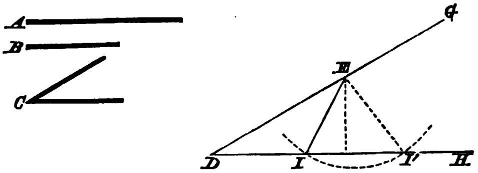
Remark. In these problems we have found that one triangle and only one can be constructed to fulfil the conditions given. In other words that with these *data* the triangle is *determinate.* Also we notice that in each case *three* elements in the triangle are *data* or given. We have given either the three sides, or two angles and the side adjacent to both, or two angles and a side opposite to one, or two sides and the included angle. And these cases correspond to the theorems proved above of the equality of triangles. For if *only one* triangle can be constructed so as to have its sides equal to three given lines, it is clear that if two triangles have the three sides of the one equal to the three sides of the other, these triangles must be identical, or be equal in all respects. And a similar remark may be made on the other cases we have considered.

But there are cases in which the data may be insufficient to determine the triangle. For example, if only two sides are given, an indefinite number of different triangles may be constructed to have these sides. Or if the three angles are given, their sum being equal to two right angles, an indefinite number of triangles may be constructed to have these three angles. And again it may be impossible to construct the triangle with the given data, as has been already shewn. In some cases moreover the solution is *ambiguous, that is, there may be more than one triangle* which fulfils the given conditions. The following is an important instance of this, and is usually called *the ambiguous case*, spoken of in Theorem 19.

PROBLEM 9.

To construct a triangle having given two sides and an angle opposite to one of them.

Let A, B be the given sides, C the angle to be opposite to the side B.



Take an angle $GDH = C$, take $DE = A$, and with centre E and radius $= B$ describe a circle. If I is one of the points in which this circle meets the line DH, by joining EI we obtain a triangle which fulfils the given conditions.

But several cases may arise.

Let the given angle be acute, as in the figure.

Then:

(1) If B is less than the perpendicular from E on DH, the circle would not meet DH, and the triangle would be *impossible.*

(2) If B is equal to the perpendicular, the circle would meet DH at the foot of the perpendicular, and there would be *one triangle, right-angled,* which fulfils the given conditions.

(3) If B is greater than the perpendicular but less than DE, then the circle will meet DH in two points I, I' as in the figure on the same side of D, and there will be *two triangles* EDI, EDI' which fulfil the given conditions.

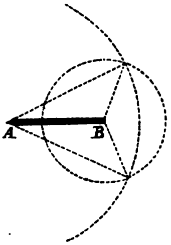
(4) If B is greater than DE, the circle will meet DH in two points on the opposite sides of D, but one only of the triangles made by joining EI, EI' will be found to have the angle D, and the other will have the supplementary angle: that is, there will be only *one solution.*

The case of the given angle being a right angle or an obtuse angle we leave to the ingenuity of the student.

Remarks on the solution of problems.

To solve problems often requires considerable inventive power. But there is one method, which we proceed to explain, of very frequent use, and it has been already used in the problems given above. This method is called that of *the intersection of loci.* A locus is the assemblage of points, every one of which fulfils an assigned geometrical condition; the meaning of this will easily be seen by a few examples. The locus of points which are at a given distance from a given point is obviously the circle described round that point with the given distance as radius. No point inside that circle, nor outside it, can be at the given distance, and every point on that circle is at the given distance. The locus of points at a given distance from an indefinite straight line is two lines parallel to that line and at the given distance from it. Hence if a point has to be determined by two conditions we may take these conditions separately, and say, that in order to fulfil the first condition it must lie on a certain line, (*the locus* of points under the first condition), and to fulfil the second condition it must lie on another line, (the locus of points under the second condition) but *the point or points of intersection* of these loci are the only point or points which lie on both lines or loci; and hence the intersection of the loci determines the points required.

For example; a triangle is to be constructed on a given base with its sides of given lengths. Let *AB* be the base.



The two conditions are that the lengths of the two sides are given, the point sought for is the vertex: now the vertex must be at a certain distance from *A* = one of the given lengths; its locus is therefore a certain circle round *A* as centre. Similarly it must be at a certain distance from *B*; its locus is therefore another circle round *B* as centre. The points of intersection of these circles are therefore the vertices of the two equal triangles which fulfil the given conditions.

It was this reasoning that suggested the construction in Problem 5.

The following exercises on the construction and intersection of loci should be worked. Occasionally it will be found that with certain data the loci do not intersect, or the solution becomes impossible. As in the case given, it will not be difficult to see that the circles would not intersect unless any two of the sides were greater than the third side.

EXERCISES ON CONSTRUCTION OF LOCI.

Find the following loci—

1. Of a point equidistant from two given points.

2. Of a point equidistant from two given lines.

3. Of a point at a given distance from a given circle.

4. Of a point at which two equal lengths not adjacent of a given straight line subtend equal angles.

5. Of the middle point of a line drawn from a given point to meet a given straight line.

6. Of a point at a given distance from the sides of a given rectangle.

7. Of a point at which two unequal straight lines, placed at a given angle, subtend equal angles.

8. A ladder, reared against a wall, slides down: find the locus of its middle point.

EXERCISES ON INTERSECTION OF LOCI.

Some of the following problems will become impossible for certain values of the lines or angles, or position of the lines or points. These impossible cases should be carefully noticed. Some also will admit of more than one solution.

1. Find a point in a given straight line at equal distances from two given points. Construct the figures for all cases.

2. Find a point in a given straight line at a given distance from a given straight line.

3. Find a point in a given straight line at equal distances from two other straight lines.

4. On a given straight line to describe an equilateral triangle.

5. Describe an isosceles triangle on a given base, each of whose sides shall be double of the base.

6. Given base, sum of sides, and one of the angles *at the base,* construct the triangle.

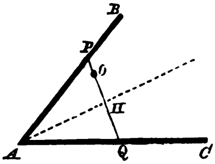
7. Given base, difference of sides, and one of the angles at the base, construct the triangle.

8. Given base, difference of sides, and difference of angles at the base, construct the triangle.

9. Find a point at a given distance from the circumference of two given circles, the distances being measured along their radii or their radii produced.

If problems cannot be solved by this method, it remains to attack them by the method, as it is called, of Analysis and Synthesis. This is not so much a method as a way of searching for a suggestion, and nothing but experience and ingenuity will here avail the student. The solution is supposed to be effected, and relations among the parts of the figure are then traced until some relation is discovered which can give a clue to the construction. Nothing but seeing examples can make this clear.

(1) It is required to draw a line to pass through a given point and make equal angles with two given intersecting lines.



Let O be the given point, AB, AC the given lines.

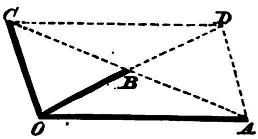
We reason as follows (*analysis*): suppose POQ were the line required, then the angle at P = angle at Q.

Therefore $AP = AQ$; therefore if we bisected the angle A, POQ would be at right angles to the bisector.

Now this is a suggestion we can work backwards from, and the construction is

Synthesis. Bisect the angle *BAC*, and let fall *OH* a perpendicular to the bisector, and let it meet the lines in *P*, *Q*, and *POQ* can then be proved to be the line required.

(2) It is required to draw from a given point three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line.



Analysis. Suppose *OA*, *OB*, *OC* were the three lines, so that *CBA* is a straight line, and *CB* $=BA$.

Then it occurs to us that if *OB* were prolonged to *D*, making $BD = OB$, then *CD* and *DA* would be respectively parallel and equal to *OA* and *OC*; (for the triangle *CBD*, *ABO* would be equal in all respects, and *CD* would be parallel to *OA*, as might be shewn), and that the sides of the triangle *DOA* are respectively equal to *OA*, *OC* and $2OB$. Hence the construction is suggested.

Synthesis. Make a triangle *DOA* whose sides are *OA*, *OC*, and $2OB$; complete the figure, by drawing *DC*, *OC*, parallel to *OA*, *AD*; and the other diagonal *ABC* will be the line required. For it may be shewn that $AB = BC$.

The student must not be surprised if he finds problems of this class difficult. For there is nothing to point out which of the many relations of the parts of the figure are to be followed up in order to arrive at the particular relation which suggests the construction. It is not easy to see what is to suggest the producing of *OB* to *D* as in the figure.

Subjoined are a few problems of no great difficulty, *which may* be solved by this method.

PROBLEMS.

1. On a given straight line to describe a square. (See Def. 30, p. 55.)

2. From a given point without a given straight line to draw a line making an angle with the line equal to a given angle.

3. Through a given point draw a line such that the part of it intercepted between two given parallel lines may have a given length.

4. Draw through a given point between two straight lines not parallel, a straight line which shall be bisected in that point.

5. Three straight lines meet in a point, draw a straight line such that the parts of it intercepted by the three lines shall be equal to one another.

6. Place a line of given length between two intersecting lines so as to be parallel to another given line.

7. Given two concentric circles, to draw a chord of the outer circle which shall be trisected by the inner.

8. Trisect a right angle.

9. Trisect a given straight line.

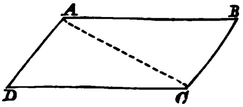
10. Given the perimeter and angles, construct the triangle.

SECTION V. PARALLELOGRAMS.

Def. 27. A four-sided figure of which the opposite sides are parallel is called a *parallelogram.*

THEOREM 21.

The opposite angles and sides of a parallelogram are equal, and the diagonal, or the line which joins its opposite angles, bisects it.



For, first, since the lines which meet in A are respectively parallel to the lines which meet in C, and are both drawn in opposite directions from those points, the angle at A = the angle at C, by Theorem 5.

Similarly the angle at B = the angle at D.

Again, if AC be joined, since the angles BAC, BCA are respectively equal to their alternate angles DCA, DAC, and AC is common, the two triangles ABC, CDA will be equal in all respects (by Theorem 16), and therefore $AB = CD$, $BC = AD$, and the area ABC = the area ADC.

COR. It is obvious that the triangle ADC is half the parallelogram $ADCB$.

Def. 28. A parallelogram which is right-angled is called a *rectangle.*

Def. 29. A parallelogram all whose sides are equal is called a *rhombus.*

Def. 30. A rectangle which has all its sides equal is called a *square.*

Def. 31. A four-sided figure which has only one pair of opposite sides parallel is called a *trapezium.*

The following exercises on parallelograms are important and should be thoroughly worked out by the student.

EXERCISES ON PARALLELOGRAMS.

1. Prove that the diagonals of a rectangle are equal to one another.

2. The diagonals of any parallelogram bisect one another.

3. The diagonals of a rhombus bisect one another at right angles.

4. Shew that a trapezium may be divided into a parallelogram and a triangle.

5. Or into a rectangle and two right-angled triangles.

6. If two straight lines be drawn bisecting one another, and their extremities be joined, the figure so formed will be a parallelogram.

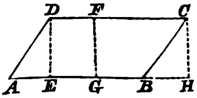
7. Given that a four-sided figure has its opposite sides equal, prove that it must be a parallelogram.

8. The straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves also equal and parallel.

SECTION VI. THE EQUIVALENCE OF FIGURES.

By *equivalent figures* are meant figures whose areas are equal although the figures may be of different shapes, and therefore not conceivable as superposed on one another. Thus a circular field may be as large as a square one, or a triangular piece of paper as large as a rectangular piece, and in such cases these figures would be called *equivalent.* The consideration of equivalent figures is an important part of Geometry.

Def. 32. The *altitude* of a parallelogram is the perpendicular distance between one side which is called the base and the side opposite to it.



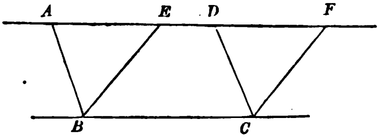
Thus in the figure at the side the perpendiculars *DE*, *FG*, or *CH*, which are equal (by Theorem 21) since *DEGF*, *DEHC* are parallelograms, are each of them the altitude of the parallelogram *ABCD*, *AB* being the base.

Def. 33. The altitude of a triangle with reference to any one side as base is the perpendicular let fall from one of its angles on the opposite side, or the opposite side produced. That side is then called the base.

THEOREM 22.

Parallelograms on the same base and between the same parallels are equivalent.

Let $ABCD$, $EBCF$ be parallelograms on the same base BC, and between the same parallels AF, BC. They shall be equivalent.



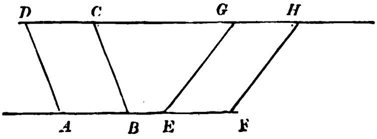
For in the triangles BAE, CDF the angle $A =$ the angle D and the angle $E =$ the angle F, by parallelism, and the side $AB =$ the corresponding side DC, and therefore the triangles are equal in area, by Theorem 16.

But if the triangle CDF is taken away from the trapezium $ABCF$, the parallelogram $ABCD$ remains: and if the triangle BAE is taken away from the same trapezium the parallelogram $EBCF$ remains. Therefore these parallelograms are equivalent.

Remark. This theorem is the fundamental theorem of equivalence of areas.

COR. 1. A parallelogram is therefore equivalent to the rectangle on the same base with the same altitude, and its area is therefore *determinate* when its base and altitude are known.

COR. 2. Parallelograms on *equal* bases and between the same parallels are equivalent.



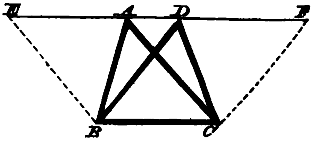
For if the base AB = the base EF, the parallelogram $AEFG$ can be conceived as superposed on the parallelogram $ABCD$, EF coinciding with AB, so that they should have the same base, and be between the same parallels, and therefore they are equivalent.

COR. 3. Parallelograms on equal bases and of the same altitude are equivalent.

For similarly they may be conceived as placed on one another so as to have the same base, since the bases are equal, and so as to be between the same parallels since the altitudes are equal; and therefore they are equivalent.

THEOREM 22.

Triangles on the same base and between the same parallels are equivalent.



Let ABC, DBC be triangles on the same base BC and between the same parallels, they shall be equivalent.

For if BE be drawn from B parallel to AC, and CF be drawn from C parallel to BD, to meet AD produced in E and F,

Then $ACBE$, $DBCF$ are parallelograms, and are equivalent to one another, by the theorem that precedes.

But the triangles ABC, DBC are respectively the halves of the parallelograms $ACBE$, $DBCF$.

Therefore the triangles are equivalent.

COR. 1. Triangles on equal bases and between the *same parallels* are equivalent.

COR. 2. Triangles on equal bases and of the same altitude are equivalent. Hence a triangle is of determinate area when its base and altitude are given.

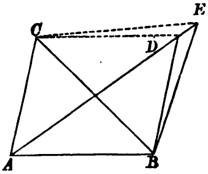
COR. 3. If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double of the triangle.

For the parallelogram *ACBE* is double of the triangle *ABC*, and therefore of any other triangle *DBC* which is on the same base, and between the same parallels.

THEOREM 23.

Equivalent triangles on equal bases will be of equal altitude.

For let the triangles *ABC*, *ABD* be equivalent triangles on equal bases, and let them be conceived as placed on the same base *AB*, and on the same side of it.



Then shall their altitudes be equal, that is, *CD* will be parallel to *AB*.

For if *CD* were not parallel to *AB*, suppose that some other line through *C*, as *CE*, meeting *AD* in *E*, were parallel to *AB*. Then the triangles *CAB*, *EAB* would be equivalent by Theorem 22.

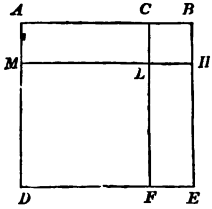
But we know that *CAB*, *DAB* are equivalent, therefore *EAD* and *DAB* would be equivalent, which is absurd.

COR. Hence equivalent triangles on the same base, and on the same side of it, must be between the same parallels.

Def. 34. A rectangle is determined when two of its adjacent sides are known. It is then said to be *contained* by its two adjacent sides, or by lines equal to them.

THEOREM 25.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the parts and twice the rectangle contained by the parts.



Let AB be divided in C.

Describe a square $ADEB$ on AB.

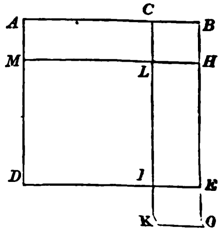
Through C draw CF parallel to AD: from BE, cut off $BH=BC$. Through H draw HLM parallel to AB.

Then it may be seen that the figures AL, CH, LE, MF are parallelograms by construction; and it is easily shewn that CH, MF are the squares on CB, AC respectively; and that AL, LE are each of them the rectangle contained by AC, CB.

Hence, since $ADEB$ is made up of these four figures, it follows that the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

THEOREM 26.

If a straight line be divided into any two parts, the square on one of the parts is less than the squares of the whole line and of the other part by twice the rectangle contained by the whole line and that other part.

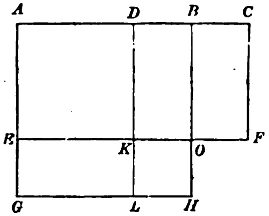


Let AB be divided in C; make the same construction as before, and on FE describe the square $FKOE$.

Then AH, HK are each of them the rectangle contained by the whole line AB and the part BC: but the *whole figure* is made up of the squares on AB and BC; and MF the square on AC is obviously less than the whole figure by the sum of the rectangles AH, HK; that is, the square on AC is less than the squares on AB and BC by twice the rectangle AB, BC.

THEOREM 27.

The rectangle contained by the sum and difference of any two lines is equivalent to the difference of the squares on those lines.



Let AB, BC be the two lines, then AC is the sum of the lines, and if BD be taken equal to BC, AD is their difference.

Draw $AE = AD$ at right angles to AC, and complete the parallelogram $AEFC$, which is therefore the rectangle contained by AC, AD, that is, by the sum and difference of AB, BC.

On AB describe the square $AGHB$, through D draw DKL parallel to AG, then KH is the square on BC.

Then it may be seen that EL, DO, BF are equal figures; but the difference of the squares on AB, BC is the figure made up of AO and EL, that is, it is equivalent to the figure made up of AO and BF, that is, to AF, which is the rectangle contained by the sum and difference of AB, BC.

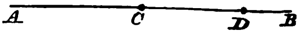
Remark. The student will begin here to suspect, what he will afterwards find to be true, that there is an intimate relation between geometry and algebra. Algebraical or analytical geometry as it is called, investigates this relation and applies it to the establishment of theorems in geometry, and will occupy him at a later stage of his mathematical studies. We shall at present use the expression AB^2, which is read 'AB squared,' only as an *abbreviation* for "the square on AB," and $AB \times AC$ as an abbreviation for "the rectangle contained by AB and AC."

These three theorems may be used to demonstrate other properties of divided lines. For example,

THEOREM 28.

If a straight line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line bisected, and the square on the line made up of the half and the part produced.

Let AB be bisected in C, and divided unequally in D. Then the squares on AD, DB will be double of the squares on AC, CD.



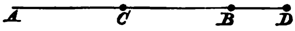
For $AD^2 = AC^2 + CD^2 + 2AC \times CD$ by Theorem 25;

and $DB^2 = CB^2 + CD^2 - 2BC \times CD$ by Theorem 26;

therefore, adding, and remembering that $AC = BC$, and that therefore $AC \times CD = BC \times CD$, we get that

$$AD^2 + DB^2 = 2AC^2 + 2CD^2.$$

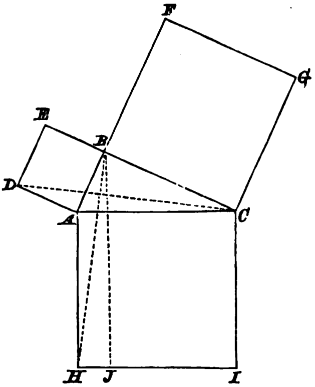
COR. If the point D is taken in AB produced, and AB is bisected in C, still $AD^2 + DB^2 = 2AC^2 + 2CD^2$.



For exactly the same proof applies to this case.

THEOREM 29.

In any right-angled triangle the square on the hypothenuse is equivalent to the sum of the squares on the sides which contain the right angle.



Let ABC be a triangle right-angled at B. Then will $AC^2 = AB^2 + BC^2$.

On AB, BC, CA describe the squares $ADEB$, $BFGC$, $CIHA$ respectively. Join CD, BH; and draw BJ parallel to AH.

Since the angles ABC, ABE, CBF are right angles, it follows that CBE, ABF are straight lines. Therefore the triangle DAC is on the same base DA, and between the same parallels DA, EC with the square $DABE$.

Therefore the triangle DAC is half the square $DABE$.

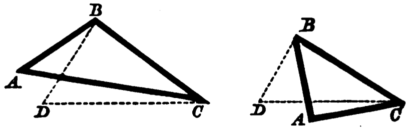
Similarly the triangle BAH is half the rectangle AJ.

But the triangles DAC, BAH are equal; for the sides DA, AC are respectively equal to BA, AH, and the contained angle $DAC =$ the contained angle BAH, each of them being a right angle together with BAC.

Therefore the rectangle $AJ =$ the square $DABE$.

Similarly it may be shown that the rectangle $CJ =$ the square $BCGF$, and therefore, since AJ and CJ make up the whole square $AHIC$, the square $AHIC$ is equivalent to the sum of the squares $ABDE$ and $BCGF$, that is, $AC^2 = AB^2 + BC^2$.

COR. 1. If ABC be an obtuse angle, or an acute angle, then AC^2 is not equal to $AB^2 + BC^2$.



For if BD is drawn at right angles to BC and equal to BA, and DC joined, then AC is greater or less than DC, according as the angle CBA is obtuse or acute by Th. 20. Therefore AC^2 is greater or less than DC^2, that is, than $DB^2 + BC^2$ or than $AB^2 + BC^2$.

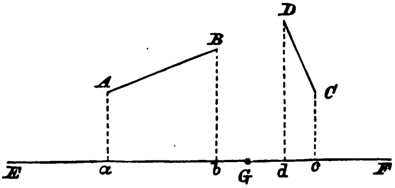
COR. 2. Hence it follows that the converse theorem holds, viz. that if the square on one side of a triangle is equal to the squares on the other two sides, the triangle is right-angled.

COR. 3. It follows that in a triangle right-angled at B,

$$AB^2 = AC^2 - BC^2 \text{ and } BC^2 = AC^2 - AB^2.$$

Def. 35. *The projection of one line on another line* is the portion of the latter intercepted between perpendiculars let fall on it from the extremities of the former.

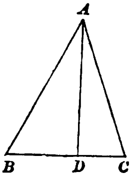
Thus the projections of AB, CD on EF are the lines ab, cd respectively.



It is clear that the line EF must be supposed indefinitely long. There could be no projection of AB on the terminated line GF.

THEOREM 30.

In any triangle the square on a side opposite an acute angle is less than the squares on the sides containing that angle by twice the rectangle contained by either of those sides and the projection on it of the other side.



Let ABC be a triangle, B an acute angle, BD the projection of AB on BC, then will

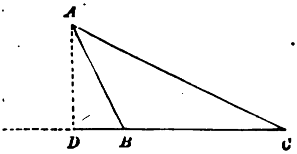
$$AC^2 = AB^2 + BC^2 - 2CB \times BD.$$

For $AC^2 = AD^2 + DC^2$ by Theorem 29,
but $AD^2 = AB^2 - BD^2$, by the same Theorem,
and $DC^2 = BC^2 + BD^2 - 2CB \times BD$ (by Theorem 26).

Therefore $AC^2 = AB^2 + BC^2 - 2\,CB \times BD$.

THEOREM 31.

In an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle by twice the rectangle contained by either of these sides and the projection on it of the other side.



Let ABC be the triangle, ABC being the obtuse angle

BD the projection of AB on BC, BC being produced backward.

Then will $AC^2 = AB^2 + BC^2 + 2CB . BD$,

for $AC^2 = AD^2 + DC^2$, by Theorem 29,

but $AD^2 = AB^2 - BD^2$,

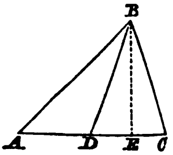
and $DC^2 = CB^2 + BD^2 + 2CB . BD$, by Th. 25,

therefore $AC^2 = AB^2 + BC^2 + 2CB . BD$.

EXERCISES ON EQUIVALENT FIGURES.

These theorems are of the greatest importance in geometry, and are largely used in geometrical theorems and especially in Trigonometry. We shall give examples of their use in establishing the two following theorems.

Ex. 1. In any triangle the sum of the squares on any two sides is double of the sum of the squares on half the base and on the line which joins the vertex to the middle point of the base.



Let AC, a side of the triangle ABC, be bisected in D; then will

$$AB^2 + BC^2 = 2AD^2 + 2BD^2.$$

Let DE be the projection of BD on AC.

Then $AB^2 = AD^2 + DB^2 + 2AD . DE$ (by Theorem 31)

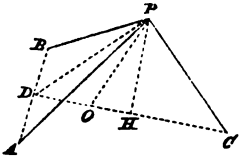
and $BC^2 = CD^2 + DB^2 - 2CD . ED$ (by Theorem 30),

therefore remembering that $AD = DC$, we obtain by addition that

$$AB^2 + BC^2 = 2AD^2 + 2DB^2.$$

Ex. 2. The locus of the point which moves so that the sum of the squares on its distances from three given *points is* constant is a circle.

Let A, B, C be the three points, P the moving point in any position.



Then $AP^2 + BP^2 + CP^2$ is to be constant.

Join AB, and bisect it in D.

Then $AP^2 + BP^2 = 2BD^2 + 2DP^2$,

and therefore $AP^2 + BP^2 + CP^2 = 2BD^2 + 2DP^2 + CP^2$.

Join DC, and let O be the point of trisection of DC nearest to D. Join OP, and let OH be its projection on OC.

Then $DP^2 = DO^2 + OP^2 + 2DO \,.\, OH$.

Therefore $2DP^2 = 2DO^2 + 2OP^2 + 4DO \,.\, OH$;

and $\qquad CP^2 = CO^2 + OP^2 - 2CO \,.\, OH$;

therefore adding these, and remembering that $CO = 2DO$,

we get $\qquad 2DP^2 + CP^2 = 2DO^2 + CO^2 + 3OP^2$,

and therefore

$$AP^2 + BP^2 + CP^2 = 2BD^2 + 2DO^2 + CO^2 + 3OP^2,$$

and since $\qquad 2BL^2 + 2DO^2 = OB^2 + OA^2$,

therefore $AP^2 + BP^2 + CP^2 = OA^2 + OB^2 + OC^2 + 3OP^2$.

Now OA, OB, OC are all independent of the position of P;

Hence if $AP^2 + BP^2 + CP^2$ is constant, OP^2 is constant, that is, OP is constant, and the locus of P is a circle described round the point O which is determined by the construction above given.

EXERCISES.

1. If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.

2. Construct a square double of a given square.

3. Construct a square equal to two, or three, or any number of given squares.

4. Divide a straight line into two parts, such that the square of one of the parts may be half the square on the whole line.

5. Given the base, area, and one of the angles at the base, construct the triangle.

6. Find the locus of a point which moves so that the sum of the squares of its distance from four given points is constant.

On the Quadrature of a Rectilineal Area.

There is one more problem which from its historical interest, and from the valuable illustrations it affords of the methods and limitations of Geometry, should find a place here. This problem is called *the quadrature of a rectilineal area*, which means the finding a square whose area is equivalent to that of any given figure which is bounded by straight lines. It gave a means of comparing any two dissimilarly shaped rectilineal figures, such as irregularly shaped fields whose boundaries were straight. In the present condition of mathematics it is not necessary, as the student will hereafter learn, but it will always be instructive.

The problem is approached by the following stages.

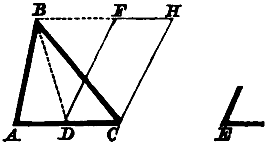
(1) To construct a parallelogram, with sides inclined at a given angle, equivalent to a given triangle.

(2) To construct *on a given straight line* a parallelogram, with sides inclined at a given angle, equivalent to a given triangle.

(3) To construct a parallelogram, with sides inclined at a given angle, equivalent to a *given rectilineal figure.*

(4) To construct a *square* equivalent to a given rectilineal figure.

(1) To construct a parallelogram, with sides inclined at an angle $= E$, equal to the triangle ABC.



Construction. Bisect AC in D, make the angle $CDF = E$, and through B draw BF parallel to AC, and draw CH parallel to DF.

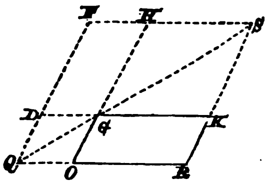
$FDCH$ will be the parallelogram required.

Proof. If BD be joined, it will be clear that the triangle BAC and the parallelogram $FHCD$ are each of them double of the triangle BDC, and therefore the parallelogram $FHCD =$ the triangle BAC and it has an angle $= E$, which was required.

It is now possible to proceed to the second stage of the problem, viz.

(2) To construct on a given straight line, a parallelogram, having a given angle, equal to a given triangle.

Let BAC be the given triangle, E the given angle as before, and let it be required to construct on the line GO a parallelogram equivalent to BAC, and having an angle E.



Construction. Construct the parallelogram *FDGH* as before, and place it so that one of its sides *GH* may be in the same straight line with *GO*.

Produce *FD*, and draw *OQ* parallel to *GD* to meet *FD* in *Q*. Join *QG*, and produce it to meet *FH* produced in *S*.

Draw *SKR* parallel to *FQ*, meeting *DG* produced in *K*, and *QO* produced in *R*.

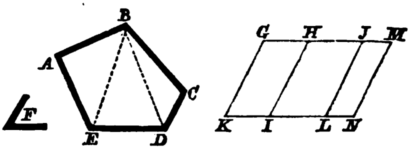
Then *GORK* is the parallelogram required.

Proof. For since the triangle *QFS* = the triangle *QRS*, and the triangles *QDG*, *GHS* = the triangles *QOG*, *GKS*; therefore the remainder, the parallelogram *FG*, = the remainder, the parallelogram *GR*, and *FG* = the given triangle *ABC*; therefore *GR* = the triangle *ABC*, and it has an angle = *E*, as is sufficiently obvious.

The third stage of the problem consists in a repeated application of these two constructions.

(3) To construct a parallelogram, having a given angle, equivalent to a given rectilineal figure.

Let *ABCDE* be the given rectilineal figure. *F*, the given angle. Divide *ABCDE* into triangles by joining *BE*, *BD*.



Construction. Construct as before a parallelogram *GHIK* = *BAE*, and having an angle at K = F.

Construct on HI a parallelogram, $HJLI = BED$, and having the angle $HIL = K$.

And construct on JL a parallelogram $JMNL = BCD$ and having the angle $JLN = F$.

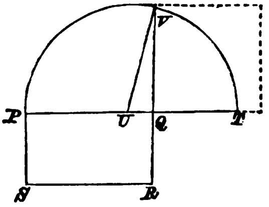
$GKNM$ will then be the parallelogram required.

Proof. For since the angle HIL = the angle K, it is therefore supplementary to HIK; and therefore (by Theorem 3) KIL is a straight line. Similarly GM and KN are straight lines, and MN is obviously parallel to GK; therefore $GKNM$ is a parallelogram, having the given angle, and it is by construction equivalent to the given rectilineal figure.

Remark. If the given figure is not rectilineal, it cannot be divided into triangles: hence by this method it is impossible to construct a square equal to a given curvilinear area. Nor can any method depending on the use of the ruler and compasses only, see p. 38, construct a square equal to some curvilinear areas, such as the circle. This is the problem of squaring the circle, the solution of which cannot be effected without the use of other instruments.

Now we are able to solve the original problem.

(4) Let it be required to construct *a square* $= ABCDE$.



Construction. By the previous construction make a *rectangle* equal to $ABCDE$, and let $PQRS$ be the rectangle so made.

Then if $PQ = QR$ the rectangle is a square; but if not, produce PQ to T, making $QT = QR$; on PQT as diameter describe a semicircle, U being the centre, and produce RQ to meet the circumference in V.

If a square be described on QV, this square will be equal to $ABCDE$.

Proof. For since PQ is the sum of PU and UQ, and QT is the difference of PU (or UT) and UQ, it follows (from Theorem 27) that the rectangle $PQ.QT = PU^2 - UQ^2$; but $PU^2 = UV^2$, and therefore $PU^2 - UQ^2 = UV^2 - UQ^2$, that is, VQ^2.

But the rectangle PQ, QT is the rectangle $PQRS$, which was made equal to $ABCDE$.

Therefore $VQ^2 = ABCDE$, and the square described on VQ is the square required.

We subjoin a few problems and theorems as miscellaneous exercises in the Geometry of angles, lines, triangles, parallelograms, and the equivalence of figures.

MISCELLANEOUS THEOREMS AND PROBLEMS.

1. Prove that the diagonals of a parallelogram bisect one another.

2. The diagonals of a rhombus bisect one another at right angles.

3. If the diagonals of a quadrilateral figure bisect one another, then the quadrilateral will be a parallelogram.

4. If the diagonals of a quadrilateral bisect one another and are equal to one another, the figure will be a rectangle.

5. If the diagonals of a quadrilateral bisect one another at right angles and are also equal, the figure will be a square.

6. If the opposite sides of a quadrilateral are equal, then the figure is a parallelogram.

7. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

8. If $ABCD$ is a parallelogram, and $AE = CF$ are cut off from the diagonal AC, then $BEDF$ will be a parallelogram.

9. If $AA' = CC'$ be cut off from the diagonal AC, and $BB' = DD'$ from the diagonal BD of a parallelogram, then will $A'B'C'D'$ be also a parallelogram.

10. If $AA' = BB' = CC' = DD'$ be cut off from the sides of the parallelogram $ABCD$ taken in order, then will $A'B'C'D'$ be also a parallelogram.

11. *ABC* is a triangle, and through *D*, the middle point of *AB*, *DE*, *DF* are drawn parallel to the sides *BC*, *AC* to meet them in *EF*. Shew that *EF* is parallel to *AB*.

12. Through a given point to draw a line such that the part of it intercepted between two parallel lines shall have a given length.

13. To describe a rhombus equal to a given parallelogram, having its side equal to the longer side of the parallelogram.

14. Shew that the diagonal of a rectangle is longer than any other line whose extremities are on the sides of the rectangle.

15. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the opposite sides; the angles made by them with the base are equal to half the vertical angle.

16. If one angle of a triangle is equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

17. *O* is any point within the triangle *ABC*, shew that *OA*, *OB*, *OC* together, are less than the sum, and greater than half the sum, of the sides of the triangle.

18. *ABC* is a triangle, *AB* greater than *BC*; *BD* bisects the base *AC*, and *BE* the angle *ABC*. Prove (1) that *ADB* is an obtuse angle; (2) that *ABD* is less than *DBC*; and (3) that *BE* is less than *BD*.

19. If two sides of a triangle be given, its area will be greatest when they contain a right angle.

20. *BCD*... are points on the circumference of a circle, *A* any point not the centre of the circle. Shew that of the lines *AB*, *AC*, *AD*... not more than two can be equal.

21. Of all triangles having the same base and area, that which is isosceles has the least perimeter.

22. Of all triangles having a given vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in that point.

23. The diagonals of a parallelogram divide it into four equivalent triangles.

24. If from any point in the diagonal of a parallelogram straight lines be drawn to the angles, then the parallelogram will be divided into two pairs of equivalent triangles.

25. *ABCD* is a parallelogram, and *E* any point in the diagonal *AC* produced. Shew that the triangles *EBC*, *EDC* will be equivalent.

26. *ABCD* is a parallelogram, and *O* any point within it, shew that the triangles *OAB*, *OCD* are together equivalent to half the parallelogram.

27. On the same supposition if lines are drawn through *O* parallel to the sides of the parallelogram, then the difference of the parallelograms *DO*, *BO* is double of the triangle *OAC*.

28. The diagonals of a parallelogram intersect in *O*, and *P* is a point within the triangle *OAB*. Prove that the difference of the triangles *APB*, *CPD*, is equivalent to the sum of the triangles *APC*, *BPD*.

29. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

30. Shew that the sum of the squares on the lines

joining the angular points of a square to any point within it is double of the sum of the squares on the perpendiculars from that point on the sides.

31. If the sides of a quadrilateral figure be bisected, and the points of bisection joined, prove that the figure so formed will be a parallelogram equal in area to half the given quadrilateral.

32. Any line drawn through the intersection of the diagonals of a parallelogram bisects the figure.

33. D is the middle point of the side AC of a triangle ACB, and any parallel lines BE, DF are drawn to meet AC, AB (or BC) in E and F, shew that EF divides the triangle into two equal areas.

34. If the sides of a triangle are 3, 4, 5 inches respectively, the triangle is right-angled.

35. The area of a rhombus is equal to half the rectangle constructed on the two diameters of the rhombus.

36. If two opposite sides of a quadrilateral are parallel, and their points of bisection joined, the quadrilateral will be bisected.

37. If two opposite sides of a parallelogram be bisected, and lines be drawn from these two points of bisection to the opposite angles, these lines will be parallel, and will trisect the diagonal.

38. The sum of the squares described on the sides of a rhombus is equal to the squares described on its diameters.

39. From the sides of the triangle ABC, AA', BB', CC', are cut off each equal to two-thirds of the side from which it is cut. Shew that the triangle $A'B'C'$ is one-third of the triangle ABC.

40. Find the locus of the vertices of triangles of equal area upon the same base.

41. Find the locus of a point, such that the sum of the squares on its distances from two given points is equal to the square on the distance between the two points.

42. Find the locus of a point such that the sum of the squares on its distances from two given points is constant.

43. Through two given points on opposite sides of a straight line draw two straight lines to meet in that line, so that the angle which they form shall be bisected by that line.

44. Through a given point draw a line such that the perpendiculars on it from two given points may be equal.

45. Find points D, E in the equal sides AB, AC of an isosceles triangle ABC, such that $BD = DE = EC$.

46. Given two points and a straight line of indefinite length, construct an equilateral triangle so that two of its sides shall pass through the given points, and the third shall be in the given straight line.

47. Construct an isosceles triangle having the angle at the vertex double of the angles at the base.

48. Bisect a triangle by a line passing through one of its angular points.

49. Bisect a triangle by a line passing through a point in one of its sides.

50. Bisect a parallelogram by a line passing through any given point.

51. Construct a triangle equal to a given quadrilateral figure.

52. Bisect a given quadrilateral figure by a line drawn from one of its angular points.

53. Bisect a given five-sided figure by a line drawn from one of its angular points.

54. Produce a given straight line to such a distance that the square on the produced part may be double of the square on the given line.

55. Produce a given straight line to such a distance that the square on the whole line may be double of the square on the given line.

56. Divide a straight line into two parts such that the square on the whole may be double of the square on one of the parts.

57. Divide a straight line into two parts such that the square on one part may be double of the square on the other.

58. From B, one of the angles of a triangle ABC, a perpendicular BD is let fall on AC. Shew that the difference of the squares on AB, BC is equal to the difference of the squares on AD, DC.

59. AC one of the sides of a triangle ABC is bisected in D: and BD joined. Shew that the squares on AB and BC together are equal to twice the square on BD, and twice the square on AD.

60. P is any point on a circle, in the diameter of which two points A, B are taken equally distant from the centre. Shew that $AP^2 + BP^2$ is constant.

61. $ABCD$ is the diameter of two concentric circles, P, Q any points on the outer and inner circles respectively. Prove that $BP^2 + CP^2 = AQ^2 + DQ^2$.

62. Prove that the squares on the diagonals of a rectangle are together equal to the squares on its sides.

63. Prove that the squares on the diagonals of any parallelogram are together equal to the squares on its sides.

64. O is the point of intersection of the diagonals of a square $ABCD$, and P any other point whatever. Prove that $AP^2 + BP^2 + CP^2 + DP^2 = 4OA^2 + 4OP^2$.

65. If a square is inscribed in a circle, the sum of the squares on the distances of any point in the circumference of the circle from the angles of the square is double of the square on the diameter of the circle.

66. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and the line so drawn are together equal to the squares of the segment adjacent to the right angle and on the hypothenuse.

67. If from the right angle C of a right-angled triangle ABC straight lines be drawn to the opposite angles of the square on AB, the difference of the squares on these two lines will equal the difference of the squares on AC and BC.

68. AB is divided into two unequal parts in C and equal parts in D; shew that the squares on AC and BC are greater than twice the rectangle $AC \times CB$ by four times the square on CD.

69. In any right-angled triangle the square on one of the sides containing the right angle is equal to the rectangle contained by the sum and difference of the other two sides.

70. In any isosceles triangle ABC, if AD is drawn from A the vertex to any point D in the base, shew that

$$AB^2 = AD^2 + BD \,.\, DC.$$

71. If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.

72. The square of the base of an isosceles triangle is double the rectangle contained by either side, and the projection on it of the base.

73. The squares on the diagonals of a quadrilateral are double of the squares on the sides of the parallelogram formed by joining the middle points of its sides.

74. Hence shew that they are also double of the squares on the lines which join the points of bisection of the opposite sides of the quadrilateral.

75. The squares on the diagonals of a quadrilateral are together less than the squares on the four sides by four times the square on the line joining the points of bisection of the diagonals.

76. In any quadrilateral figure the lines which join the middle points of opposite sides intersect in the line which joins the middle point of the diagonals.

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